

TORIC VARIETIES IN HILBERT SCHEMES

HEATHER RUSSELL

CONTENTS

1. Introduction	1
2. Finding the Boundary as a Limit Point	2
3. Toric Varieties as Fibers	4
4. Almost Toric Varieties	14
References	16

1. INTRODUCTION

Given a field K , let R be the power series ring $K[[x_1, \dots, x_n]]$. There is a natural action of $\text{Aut}(R)$ on the Hilbert scheme $\text{Hilb}^d(R)$ parameterizing ideals in R of colength d . Recall from [10] that given a smooth variety X of dimension n and a monomial ideal $I \subset R$ of finite colength, the space $U(I)$ parametrizing subschemes of X isomorphic to $\text{Spec}(R/I)$ sits naturally inside a suitable Hilbert scheme H . In [10] a flag bundle B on the tangent bundle of X and a space $Y \subset B \times U(I)$ such that Y is a fiber bundle over B with respect to the first projection map and an étale covering of $U(I)$ with respect to the second is constructed. Moreover, Y has the property that its closure in $B \times H$ is also a fiber bundle over B . Via an isomorphism φ of R and the local ring of the point of X over which the fiber lies, the fiber is isomorphic to the closure of the G -orbit of I in H where G is the subgroup of $\text{Aut}(R)$ fixing the flag over which the fiber lies via φ . The complexity of the fiber $F(I)$ of Y over B and its closure is partially measured by the measuring sequence of I as defined in [10]. In the first level of complexity, the G equivariance of $\bar{F}(I)$ forces its normalization to be a projective space. In the second level, the normalization is forced to be a toric variety. In the third level, we lump together all of the other cases. The following three sections correspond to these three levels of difficulty, each illustrating techniques for understanding F and $\bar{F}(I)$. For simplicity, we restrict ourselves to the case when $n = 2$, thus freeing the variable n for later use. Moreover, we will use the variables x

and y in place of x_1 and x_2 , to cut down on the use of subscripts. In this case it is very easy to define the measuring sequences.

Definition: Say a sequence of monomial ideals I_1, \dots, I_r of R , each having finite colength, has *measuring sequence* $(x, y^a), (x^b, y)$ denoted $m(a, b)$ where a and b are the smallest positive integers such that any automorphism of R fixing the measuring sequence also fixes the original sequence of ideals. We will say that $m(a, b) \leq m(c, d)$ if $a \leq c$ and $b \leq d$.

If the measuring sequence of I is $m(1, 1)$ then I must be a power of the maximal ideal \mathfrak{m} of R . Otherwise, up to permuting x and y , the group G is the group of automorphisms sending the ideal (x, y^2) to itself and the fiber $F(I)$ of Y over B is isomorphic to the quotient of G by the subgroup $G(I)$ of automorphisms fixing I . To avoid defining G repeatedly, we will henceforth let G be the group of automorphisms fixing (x, y^2) . The measuring sequence of the ideal I determines the group $G(I)$. More generally, if we consider the G orbit of a sequence of monomial ideals $F(I_1, \dots, I_r)$, then this space is isomorphic to the quotient of G by its subgroup $G(I_1, \dots, I_r)$ of automorphisms fixing the I_j 's. The group $G(I_1, \dots, I_r)$ is likewise determined by the measuring sequence of the I_j 's.

Given a monomial ideal $I \in R$ let $\Gamma(I)$ denote the staircase associated to I . Then $\Gamma(I)$ consists of the unit squares tiling the plane with lower left hand corner giving the exponent vector for a monomial in the complement of I . In characteristic zero, the measuring sequence of ideals I_1, \dots, I_r is of the form $m(a, b)$ where a is the largest number of boxes by which two consecutive rows differ in any of the $\Gamma(I_j)$'s and b the largest number by which two consecutive columns differ.

2. FINDING THE BOUNDARY AS A LIMIT POINT

If the sequence of ideals I has measuring sequence at most $m(2, 1)$, then $F(I)$ is a single point. If it is $m(3, 1)$ or $m(2, 2)$, then the space $F(I)$ is an affine line and its closure $\bar{F}(I)$ is a \mathbb{P}^1 . Thus the boundary is a single point. In this section we give an algorithm for finding the boundary point simply by shifting boxes of the staircases Γ_i .

By Lemma 3.3 of [10], if the measuring sequence of I is $m(3, 1)$ (respectively $m(2, 2)$) a set of coset representatives for $G/G(I_1, \dots, I_r)$ are given by automorphism of the form $g(t)$ where $g(t)(x) = x + ty^2$ and $g(t)(y) = y$ (respectively $g(t)(x) = x$ and $g(t)(y) = y + tx$). Thus the boundary point is the limit as t goes to infinity of $g(t)(I)$ and can be found by Proposition 2.1 below.

Definition: If the characteristic p of K is positive, let \prec be the total ordering on the set of non-negative integers defined by $a \prec b$ if either $\text{ord}_p(a) > \text{ord}_p(b)$ or $\text{ord}_p(a) = \text{ord}_p(b)$ and $a < b$. Otherwise, let it be the usual ordering. Given a finite collection of non-negative integers T , let f be the injective map from T to the set of non-negative integers sending each element of T in ascending order with respect to \prec to the smallest integer divisible by a greater or equal power of p . Define the p -shift T' of T to be the image of f .

Lemma 2.1. *Let T be a finite collection of non-negative integers. The limit $V(T)$ as t goes to infinity of the subspaces $V(T, t)$ of $K[x]$ generated by the polynomials $(x + t)^e$ for exponents $e \in T$ is spanned by the monomials x^e for exponents $e \in T'$ where T' is the p -shift of T .*

Proof: Let $r = |T|$ and assume that the lemma holds for all smaller cardinalities. Suppose that T' is the set of integers from 0 to $r - 1$. Let M be the span of wedge products of monomials $\wedge^r(K[x])$ such that for some power of p there are more exponents of the monomials divisible by that power than elements of T' divisible by that power. Then M is fixed by $g(t)$ for any t . Thus so is the limit as t goes to infinity of the images of the projections of the spaces $V(T, t)$ in $\mathbb{P}(\wedge^r(K[x])/M)$. The projection map is well-defined because the coefficient of the wedge product of monomials with exponent in T is non-zero in the image of $V(T, t)$ in $\mathbb{P}(\wedge^r(K[x]))$. Since the only $g(t)$ -fixed point in $\mathbb{P}(\wedge^r(K[x])/M)$ is the wedge product of monomials with exponent in T' , it follows that the coefficient of the wedge product of monomials with exponent in T' in $\mathbb{P}(\wedge^r(K[x]))$ is non-zero and has the highest degree in t . Thus the limit $V(T)$ is as claimed.

Suppose that T' is not the set of integers from 0 to $r - 1$. Let $f : T \rightarrow T'$ be the map taking a label of a box to the label of the box it shifts to as in the definition of p -shift above. Let T_1 and T_2 be the compliments in the preimage under f of the compliments of the largest element of T' and the largest element of T' with the smallest power of p dividing it. Then

$$T'_1 + T'_2 = T'$$

and thus

$$V(T_1) + V(T_2) \subset V(T).$$

It follows from the inductive hypothesis applied to T_1 and T_2 that $V(T)$ is as claimed.

Proposition 2.1. *Let $I \subset R$ be a monomial ideal of finite colength. Let h be a monomial not divisible by x_i . For each $t \in K$, let $g(t)$ be the*

element of $\text{Aut}(R)$ with

$$g(t)(x_j) = x_j$$

for $j \neq i$ and

$$g(t)(x_i) = x_i + th.$$

Let J be the flat limit of ideals

$$\lim_{t \rightarrow \infty} g(t)(I).$$

Then J is characterized by the following. Let T be the set of integers a such that a basis of M is given by monomials of the form $x_i^a h^{d-a} f$ for $a \in T$ and f a monomial of minimal degree. Then the corresponding graded piece of J has a basis of monomials of the form $x_i^a h^{d-a} f$ for $a \in T'$.

Proof: Since f and h are invariant under $g(t)$, the proof reduces to Lemma 2.1.

3. TORIC VARIETIES AS FIBERS

If the measuring sequence is $m(4, 1)$ or $m(2, 3)$ by Lemma 3.3 [10], coset representatives for $G/G(I_1, \dots, I_r)$ are given by the set of automorphisms $g(a, b)$ with $g(a, b) = x + ay^2 + by^3$ and $g(a, b)(y) = y$ and the set of automorphisms $h(a, b)$ with $h(a, b)(x) = x + ay^2$ and $h(a, b)(y) = y + bx$ respectively where a and b range over K . Hence $F(I_1, \dots, I_r)$ is an affine plane with a and b as coordinates. Its closure, being equivariant under the action of G is equivariant under the action of scaling x and y . One can find the fan of the normalization of $\bar{F}(I_1, \dots, I_r)$ as follows. Let V_j be the quotient of two monomial ideals between which all ideal in $F(I_j)$ are sandwiched. Then $F(I_j)$ has an embedding in the Grassmanian of subspaces of V_j of the appropriate dimension which can in turn be embedded in projective space by Plücker coordinates. Taking the coordinates to correspond to wedge products of monomials, the coordinate functions will be monomials in a and b [10]. Since the product of these projective spaces for each j can be embedded into one big projective space by the Segre embedding, coordinate functions embedding $F(I_1, \dots, I_r)$ in this big projective space are given by products of the coordinate functions embedding each of the $F(I_j)$'s, one for each j . Since these will also be monomials, we can plot the exponent vectors (m, n) of each coordinate function $a^m b^n$ and take the convex hull H . The cone covering the plane with a ray normal to each edge of H is a fan $\Delta(I_1, \dots, I_r)$ for the normalization $F(I_1, \dots, I_r)$ in the sense of [6].

Definition: Given a sequence of ideals (I_1, \dots, I_r) with measuring sequence $m(4, 1)$ or $m(3, 2)$, say that the fan $\Delta(I_1, \dots, I_r)$ constructed above is the *standard fan* for $F(I_1, \dots, I_r)$.

Suppose one wanted to find the standard fan for the space $F(I(4)^m)$. The ideals in the G orbit of $I(4)^m$ are sandwiched between $I(2)^m$ and $I(2)^{2m}$. Let x have weight 2 and y have weight 1. Then the $2m^2$ polynomials of the form $g(a, b)(x^c)x^d y^e$ for $4c + 2d + e$ equal to either $4m$ or $4m + 1$ and $c > 0$ span $I(4)^m$ as a subspace of $I(2)^m/I(2)^{2m}$. Let M_m be the matrix with rows corresponding to these generators arranged in increasing order first by c and then by d , columns corresponding to the monomials spanning $I(2)^m/I(2)^{2m}$ arranged in order of descending weight and descending powers of y , and entries given the coefficient of the column's monomial in the row's polynomial. Coordinate functions embedding $F(I(4)^m)$ into projective space are given by the determinants of the maximum minors of M_m . With no additional insight this becomes a hefty task as m becomes large. However, with the help of a few observations we will find the standard tori for $F(I(4)^m)$ without taking a single determinant. Letting x have weight $(1, 0)$, y have weight $(0, 1)$ respectively, a have weight $(1, -2)$ and b have weight $(1, -3)$. Then $g(a, b)(x)$ is homogenous and thus the wedge product of the $2m^2$ polynomials are homogenous and so the sum of the weights of the determinant of a maximum minor and the monomials corresponding to the columns is constant. Thus, the powers of a and b in the determinant of a minor can be read off easily, but it can be difficult to tell even whether the coefficient is zero or not. To find the standard fan of $I(4)^m$, it is only necessary to find the vertices of the convex hull of the exponent vectors of the non-zero coordinate functions. These vertices correspond to the monomial ideals in $F(I(4)^m)$ in the sense that the one non-zero coordinate of the monomial ideal maps to the corresponding vertex. Thus we need only check the coefficients of coordinates corresponding to monomial ideals, but this may still be unnecessarily difficult. For example, the coefficient of the coordinate function corresponding to the ideal

$$I(\underbrace{1, \dots, 1}_m, 2, \underbrace{1, \dots, 1}_{m-1})$$

is the determinant of the $2m^2 \times 2m^2$ matrix which can be described as follows. Divide up the rows and columns as follows. Put the first $2m - 1$ rows together, then the next $2n - 1$ rows together, then the next $2m - 3$ rows together, then the next $2n - 3$ rows together and so on, alternately keeping the number of rows the same and decreasing

1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0
1	0	0	0	0	0	2	2	0	0	0	1	2	1	0	0	0	0
0	1	0	0	0	0	0	2	2	0	0	0	1	2	1	0	0	0
0	0	1	0	0	0	0	0	2	2	0	0	0	1	2	0	0	0
0	0	0	0	0	0	1	0	0	0	0	2	2	0	0	1	2	0
0	0	0	0	0	0	0	1	0	0	0	0	2	2	0	0	1	0
0	0	0	0	0	0	0	0	1	0	0	0	0	2	2	0	0	0
0	0	0	0	0	0	1	0	0	0	0	3	3	0	0	3	6	1
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	3	3	3

the number of rows by two. Divide the columns up similarly except that the first group of columns will have one extra column and the last group will have one less and hence no columns. Fill in the rectangles as follows. In the $(2n - 1)^{\text{st}}$ and $(2n)^{\text{th}}$ row of rectangles, the entry in the i^{th} and j^{th} column of the k^{th} rectangle is

$$\binom{n}{k+1-n} \binom{k+1-n}{j-i} \text{ and } \binom{n}{k-n} \binom{k-n}{j-i}$$

We will adopt the following notation. Given an ideal I with measuring sequence $m(4, 1)$, let $I^+(m, n)$ (respectively $I^-(m, n)$) be the ideal corresponding to the cone just clockwise (respectively counterclockwise) to the ray through (m, n) in the standard fan $\Delta(I)$. We extend this definition to ideals with smaller measuring sequence in the natural way. Similarly, if I has measuring sequence $m(3, 2)$ we let $I_+(m, n)$ (respectively $I_-(m, n)$) be the ideal corresponding to the cone just clockwise (respectively counterclockwise) to the ray through (m, n) in the standard fan $\Delta(I)$. and we extend this definition to ideals with smaller measuring sequence in the natural way.

Proposition 3.1. *Given an ideal I with measuring sequence $m(4, 1)$ or $m(3, 2)$, the standard fan $\Delta(I)$ has lower left quadrant corresponding to the affine plane $F(I)$ with the negative x and y axes corresponding to the line $a = 0$ and $b = 0$ respectively. In the first case*

$$I^+(-1, 0) = I^-(0, 1) = \lim_{t \rightarrow \infty} g(0, t)(I),$$

$$I^-(0, -1) = \lim_{t \rightarrow \infty} g(t, 0)(I),$$

$$I^+(0, 1) = \lim_{t \rightarrow \infty} g(t, 0)I^-(0, 1),$$

and, if the characteristic of K is not 2, then

$$I^-(0, -1) = I^+(1, 2)$$

and

$$I^-(1, 2) = \lim_{t \rightarrow \infty} g(0, t)I^+(1, 2).$$

In the second case,

$$I_+(-1, 0) = I_-(0, 1) = \lim_{t \rightarrow \infty} h(0, t)(I),$$

$$I_-(0, -1) = I_+(1, 0) = \lim_{t \rightarrow \infty} h(t, 0)(I),$$

$$I_+(0, 1) = \lim_{t \rightarrow \infty} h(t, 0)I_-(0, 1),$$

and

$$I_-(0, 1) = \lim_{t \rightarrow \infty} h(0, t)I_+(0, 1).$$

Proof: One can verify by hand that

$$(x, y^3)^+(-1, 0) = (x, y^3)^-(0, 1) = (x, y^3),$$

$$(x, y^4)^-(0, -1) = (x, y^4)^+(1, 2) = (x^2, y^2),$$

$$(x, y^3)_+(-1, 0) = (x, y^3)_-(0, 1) = (x, y^3),$$

$$(x^2, y)_-(0, -1) = (x^2, y)^+(1, 0) = (x^2, y).$$

Except for the ideal in the second line in characteristic 2, these ideals are not invariant under the action of G . Thus the proposition follows.

See [4] and [10] for the relevant definitions in the following corollary.

Corollary 3.1. *The Semple bundle $F(4)$ is isomorphic to the alignment correspondence $C((x, y^2), (x, y^3), (x^2, xy, y^5))$.*

Proof : Recall that the Semple bundle $F(4)$ is an compactification of the space of curvilinear 3-jets on X with the property that it is a fiber bundle over the projectivized tangent bundle of X with fiber equivariant under the action of G . Moreover, the fibers are smooth and have two boundary divisors with self-intersection 0 and -3 respectively, as one can check from the Chow ring of $F(4)$ computed in [4]. Thus, they are toric varieties and giving the interior of the fiber in a manner compatible with the standard fans constructed before, the only possible fan for the fiber of $F(4)$ is the fan for $F((x, y^3), (x^2, xy, y^5))$.

Proposition 3.2. *Given an ideal I with measuring sequence $m(4, 1)$ (respectively $m(3, 2)$), let $I[n, m]$ be the limit as t tends to infinity of $g(at^n, bt^m)(I)$ (respectively $h(at^n, bt^m)$). As $\frac{a^n}{b^m}$ varies through \mathbb{C}^* , $I[n, m]$ runs over the torus orbit corresponding to the ray through (m, n) in the standard torus $\Delta(I)$. The limits as a and b tend to 0 of $I[m, n]$ are $I^+(m, n)$ and $I^-(m, n)$ (respectively $I_+(m, n)$ and $I_-(m, n)$). In particular, giving x weight $n - m$ (respectively $n + m$) and y weight $2n - 3m$ (respectively $n + 2m$), each graded piece of the monomial ideal $I^+(m, n)$ (respectively $I_+(m, n)$) has the same dimension as the respective graded piece of the monomial ideal $I^-(m, n)$ respectively $I_-(m, n)$.*

Proof: By the construction of the standard torus $\Delta(I)$, if there is a ray in $\Delta(I)$ through (m, n) , the limit of a path (at^n, bt^m) with $ab \neq 0$ in the affine coordinates of $F(I)$, as t goes to infinity, lands in the torus orbit corresponding to that ray, the points of the torus orbit, being in bijection with the ratios $\frac{a^n}{b^m}$. Thus as a and b tend to 0, we get the two ideals $I^-(m, n)$ and $I^+(m, n)$ (respectively $I_+(m, n)$ and $I_-(m, n)$). The point (at^n, bt^m) corresponds to the ideal $g(at^n, bt^m)(I)$ (respectively $h(at^n, bt^m)(I)$). Giving a weight $-n$ and b weight $-m$, the elements of $I[m, n]$ are the coefficients of the highest powers of t in elements of $g(at^n, bt^m)(I)$ (respectively $h(at^n, bt^m)(I)$). Thus $I[m, n]$ is generated by homogenous elements. Moreover giving x weight $n - m$ (respectively $n + m$), y weight $2n - 3m$ (respectively $n + 2m$), and keeping the weights of a and b the same, $g(at^n, bt^m)(I)$ (respectively $h(at^n, bt^m)(I)$) is generated by homogenous elements. Thus $I[m, n]$, being already homogenous in a and b is homogenous in x and y with respect to the weights. Therefore the dimensions of the graded pieces of the two limits $I^-(m, n)$ and $I^+(m, n)$ (respectively $I_+(m, n)$ and $I_-(m, n)$) with respect to the weights on x and y are the same as the dimensions of the respective graded pieces of $I[m, n]$.

Corollary 3.2. *Given a monomial ideal I with measuring sequence $m(4, 1)$ (respectively $m(3, 2)$) if two monomial ideals I_1 and I_2 correspond to boundary points of $F(I)$ and the quotient of the product of*

monomials in there respective complements is $\frac{x^c}{y^d}$ with $c > 0$, then the ray r through the point $(2c - d, 3c - d)$ (respectively $(2c - d, d - c)$) lies in the convex cone bounded by the two cones in the standard fan $\Delta(I)$ corresponding to I_1 and I_2 .

Proof: If I_1 and I_2 correspond to adjacent cones in $\Delta(I)$, then by Proposition 3.2, r is the ray between them. The rest follows from elementary properties of medians.

Proposition 3.3. *Given ideals I_1 , I_2 , and I_3 with $I_1 I_2 \subset I_3$ then for any point $P \in F(I_1, I_2, I_3)$, we have $\pi_1(P)\pi_2(P) \subset \pi_3(P)$ where π_i is projection to $F(I_i)$.*

Proof: Let V_1 , V_2 , and V_3 be quotients of ideals between which the ideals in the G orbits of I_1 , I_2 , and I_3 lie respectively. Consider the incidence correspondence

$$\Gamma = \{(\alpha_1, \alpha_2, \alpha_3, (a_1, a_2, a_3)) \in V_1 \times V_2 \times V_3 \times \bar{F}(I_1, I_2, I_3) : \alpha_i \in a_i, \alpha_1 \alpha_2 = \alpha_3\}.$$

Then the set

$$\{(\alpha_1, \alpha_2, \alpha_3, (a_1, a_2, a_3)) \in V_1 \times V_2 \times V_3 \times F(I_1, I_2, I_3) : \alpha_i \in a_i\}$$

is contained in Γ and hence since so is its closure

$$\{(\alpha_1, \alpha_2, \alpha_3, (a_1, a_2, a_3)) \in V_1 \times V_2 \times V_3 \times \bar{F}(I_1, I_2, I_3) : \alpha_i \in a_i\}.$$

Given a sequence of non-negative integers,

$$s = n_1, \dots, n_r,$$

we will let $I(s)$ denote the ideal

$$I(s) = (x^r, x^{r-1}y^{n_1}, \dots, y^{n_1+\dots+n_r}).$$

We will use the following two observations repeatedly without mentioning so explicitly.

1.

$$\text{col}(I(n_1, \dots, n_r)) = \sum_{i=1}^r (r+1-i)n_i.$$

2. $I(n_1, \dots, n_r)I(m_1, \dots, m_k) = I(l_1, \dots, l_{r+k})$ where

$$l_i = \min_{\alpha+\beta=i} \left(\sum_{j=1}^{\alpha} n_j = \sum_{j=1}^{\beta} m_j \right).$$

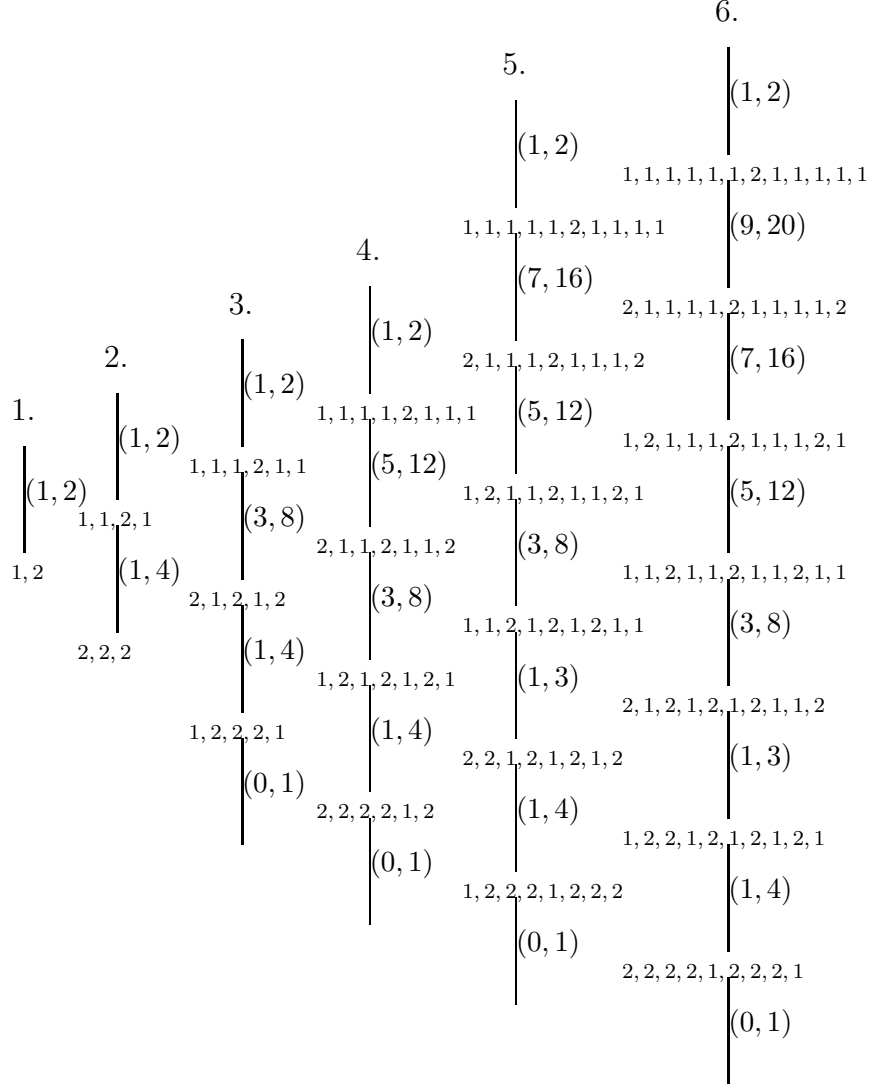


FIGURE 2. Boundary Components

We will use the notation

$$I^{\pm}(r, (m, n)) = ((x, y^4)^r)^{\pm}(m, n).$$

From the results proved in this section, we will now demonstrate the following eight claims.

1. $I^-(n, (1, 0)) = I^+(n, (1, 2)) = (x^2, y^2)^n$.

$$2. I^-(n, (1, 2)) = I(\underbrace{1, \dots, 1}_n, \underbrace{2, 1, \dots, 1}_{n-1}) = I^+(n, (2n-3, 4n-4))$$

3. Letting $s_1 = 1, 2$, $s_2 = 2, 2, 2$ and $s_3 = 3, 2, 2, 2$ then

$$I^+(3q+r, (-1, 0)) = I(s_r, \underbrace{s_3, \dots, s_3}_q) = I(s_r)I(s_3)^q$$

for $1 \leq r \leq 3$.

4. Letting $s_1 = 2, 2, 1$, $s_2 = 2, 2, 2$ and $s_3 = 2, 1, 2$,

$$I^+(2n+1, (0, 1)) = I^-(2n+1, (1, 4)) = I(1, 2, \underbrace{s_1, s_2, s_3, \dots}_n)$$

and

$$I^+(2n, (0, 1)) = I^-(2n, (1, 4)) = I(\underbrace{s_2, s_3, s_1, \dots}_n).$$

5. If $\lfloor \frac{n}{2} \rfloor \leq k \leq n-1$ then

$$I^-(n, (2k-1, 4k)) = I(\underbrace{1, \dots, 1}_{n-k-1}, \underbrace{2, 1, \dots, 1}_{k-1}, \underbrace{2, 1, \dots, 1}_{k-1}, \underbrace{2, 1, \dots, 1}_{n-k-1}).$$

6. If $\lfloor \frac{n-1}{2} \rfloor \leq k \leq n-2$ then

$$I^+(n, (2k-1, 4k)) = I(\underbrace{1, \dots, 1}_{n-k-2}, \underbrace{2, 1, \dots, 1}_k, \underbrace{2, 1, \dots, 1}_k, \underbrace{2, 1, \dots, 1}_{n-k-2}).$$

7. Letting $s_1 = 2, 2, 1, 2, 1, 2, 1, 2$ and $s_2 = 2, 1, 2, 1, 2, 1, 2, 1$, then

$$I^+(n, (1, 4)) = I^-(n, (1, 3)) =$$

$$\begin{cases} I(\underbrace{s_1, s_2, \dots}_m) & \text{if } n = 5m, \\ I(1, 2, \underbrace{s_2, s_1, \dots}_m) & \text{if } n = 5m+1, \\ I(1, 1, 2, 1, \underbrace{s_1, s_2, \dots}_m) & \text{if } n = 5m+2, \\ I(2, 1, 2, 1, 2, \underbrace{s_2, s_1, \dots}_m) & \text{if } n = 5m+3, \text{ and} \\ I(1, 2, 1, 2, 1, 2, 1, \underbrace{s_1, s_2, \dots}_m) & \text{if } n = 5m+4 \end{cases}$$

8. Letting $s_1 = 2, 1, 2, 1, 2$, $s_2 = 1, 2, 1, 2, 1$, $s_3 = 2, 1, 2, 1, 1$, $s_4 = 1, 2, 1, 1, 2$ and $s_5 = 2, 1, 1, 2, 1$, then

$$I^+(n, (1, 3)) = I^-(n, (3, 8)) =$$

$$\begin{cases} I(\underbrace{s_1, s_2, s_3, s_4, s_5, \dots}_m) & \text{if } n = 3m, \\ I(1, 2, \underbrace{s_2, s_5, s_3, s_1, s_4, \dots}_m) & \text{if } n = 3m + 1, \text{ and} \\ I(1, 1, 2, 1, \underbrace{s_3, s_1, s_4, s_2, s_5, \dots}_m) & \text{if } n = 3m + 2 \end{cases}$$

The first item holds for $n = 1$ by Proposition 3.1. By Proposition 3.3 and induction on n , it holds for all n .

By Proposition 3.2, $I^-(n, (1, 2))$ contains $I^-(n-1, (1, 2))I^-(1, (1, 2))$. By induction on n , this is the ideal

$$I(\underbrace{1, \dots, 1}_n, 2, \underbrace{1, \dots, 1}_{n-2}, 2).$$

Proposition 3.2 and (1) imply that $I^-(n, (1, 2))$ contains y^{2n+1} . Checking colengths, this forces

$$I^-(n, (1, 2)) = I(\underbrace{1, \dots, 1}_n, 2, \underbrace{1, \dots, 1}_{n-1}).$$

The third item follows from Proposition 3.3 after checking the first three ideals by hand via Proposition 3.1 and checking that all of the ideals have the correct colength. This can be done by induction, checking the cases of the three possible r 's separately.

For $n \leq 3$, (4) follows from Proposition 3.1. Then by induction on n and checking colengths, by Proposition 3.3 we have

$$I^+(n, (0, 1)) = I^+(n-2, (0, 1))I^+(2, (0, 1)) + I^+(n-3, (0, 1))I^+(3, (0, 1))$$

which is the claimed ideal.

By Proposition 3.1, in all characteristics except possibly 2,

$$I^+(1, (0, -1)) = I^-(1, (1, 2)) = I(1, 2).$$

Since the polytope in the construction of a standard fan for positive characteristic must be contained in the corresponding polytope in characteristic 0, this last result holds also in characteristic 2 and the boundary of $F(I(4))$ is as described in Figure 3 in all characteristics.

We will prove (5) and (6) by induction on n . For $n \geq 2$ by Proposition 3.3 we have (5) for $\lceil \frac{n+1}{2} \rceil \leq k \leq n-2$ and (6) for $\lceil \frac{n-1}{2} \rceil \leq k \leq n-3$ since for

$$\left\lceil \frac{n+1}{2} \right\rceil \leq k < n-2$$

we have

$$\begin{aligned} I^-(n, (2k-1, 4k)) &= I^+(n, (2k-3, 4k-4)) = \\ &= I^-(n-1, (2k-1, 4k))I^-(1, (2k-1, 4k)) + \end{aligned}$$

$$\begin{aligned}
& I^-\left(\left\lceil \frac{n}{2} \right\rceil, (2k-1, 4k)\right) I^-\left(\left\lceil \frac{n+1}{2} \right\rceil, (2k-1, 4k)\right) \\
&= I(\underbrace{1, \dots, 1}_{n-k-1}, 2, \underbrace{1, \dots, 1}_{k-1}, 2, \underbrace{1, \dots, 1}_{k-1}, 2, \underbrace{1, \dots, 1}_{n-k-1}).
\end{aligned}$$

Thus for $n \geq 3$,

$$I^-(n, (2n-5, 4n-8)) = I(1, 2, \underbrace{1, \dots, 1}_{n-3}, 2, \underbrace{1, \dots, 1}_{n-3}, 2, 1).$$

By Proposition 3.3, $I^+(n, (2n-5, 4n-8))$ is contained in $I^+(1, (2n-5, 4n-8))I^+(n-1, (2n-5, 4n-8))$. Proposition 3.2 then forces $I^+(n, (2n-5, 4n-8))$ to be as claimed. By Proposition 3.3 the points corresponding to 2-dimensional cones in the standard fan $\Delta((x, y^4)^n)$ lying in the convex cone bounded by the rays through $(1, 2)$ and $(2n-5, 4n-8)$ correspond to ideals contained in

$$J = I^-(n-1, (1, 2))I^-(1, (1, 2)) = I(\underbrace{1, \dots, 1}_n, 2, \underbrace{1, \dots, 1}_{n-2}, 2).$$

The cones corresponding to the ideals $I^-(n, (1, 2))$ and $I^+(n, (2n-5, 4n-8))$ lie in the cone bounded by the rays through $(1, 2)$ and $(2n-5, 4n-8)$. Suppose there is another such cone corresponding to an ideal generated by $x^e y^f$ over J . Since $I^-(n, (1, 2))$ and $I^+(n, (2n-5, 4n-8))$ are generated over J by y^{2n+1} and x^{2n-1} the inequalities

$$\frac{2(2n-1-e)-f}{3(2n-1-e)-f} > \frac{2e-2n-1+f}{3e-2n-1+f} > \frac{1}{2}$$

follow from Corollary 3.2. Thus it follows that $e+f=2n$ and $e \geq f$. However, this implies that $x^e y^f$ is already contained in J . Therefore we have arrived at a contradiction and by Corollary 3.2, $I^+(n, (2n-5, 4n-8)) = I^-(n, 2n-3, 4n-4)$ and $I^-(n, (1, 2)) = I^+(n, 2n-3, 4n-4)$. If $n = 2m+1$, the ideal $I^-(n, (2m-1, 4m))$ cannot be deduced from Proposition 3.3 alone, containment in $I^-(n-1, (2m-1, 4m))I^-(1, (2m-1, 4m))$ together with knowledge of $I^+(n, (2m-1, 4m))$ and Proposition 3.2 force it to be as claimed.

From what we have shown so far, it follows that the boundary of $\bar{F}((x, y^4)^n)$ for $n \leq 4$ is as depicted in Figure 3. Each vertical line segment represents the boundary divisor corresponding to the 1-dimensional cone passing through the lattice pointed as marked. The ideals in between the line segments represent the intersections of the corresponding boundary divisors.

We will now find the boundary of $\bar{F}((x, y^4)^5)$. By Proposition 3.3 $I^+(5, (1, 4))$ is contained in $I(1, 1, 2, 1, 2, 1, 2, 1, 2)$. Thus Proposition 3.2 forces $I^+(5, (1, 4)) = I(2, 2, 1, 2, 1, 2, 1, 2)$. In the same manner that

Corollary 3.2 was used to deduce $I^-(n, (1, 2)) = I^+(n, (2n - 3, 4n - 4))$, it can be used to deduce that $I^+(5, (1, 4)) = I^-(5, (1, 3))$ and $I^-(5, (3, 8)) = I^-(5, (1, 3))$, completing the diagram for $\bar{F}((x, y^4)^5)$.

It remains to verify the last two items. These items follow from induction on n and Proposition 3.3 since

$$I^+(n, (1, 4)) = I^+(n-2, (1, 4))I^+(2, (1, 4)) + I^+(n-5, (1, 4))I^+(5, (1, 4))$$

and

$$I^+(n, (1, 3)) = I^+(n-3, (1, 3))I^+(3, (1, 3)) + I^+(n-5, (1, 3))I^+(5, (1, 3)).$$

Thus we have demonstrated our eight claims, proving the boundaries of the spaces $\bar{F}((x, y^4)^n)$ to be as in Figure 3 for $n \leq 6$. We stop here, not because our techniques do not suffice for larger n , but because we have illustrated them all.

4. ALMOST TORIC VARIETIES

In the last section we developed an arsenal of techniques for understanding the spaces $F(I)$ for ideals I with certain measuring sequences. The basic idea was to use the structure of the normalization of $F(I)$ as a toric variety together with the correspondence of the points of $F(I)$ as ideals. In this section we take a baby step towards developing a similar arsenal for dealing with other measuring sequences by considering a few examples. In these examples we consider some spaces $\bar{F}(I_1, \dots, I_r)$ in which the I_j 's have measuring sequence at most $m(5, 1)$. These spaces will be three dimensional, with a two dimensional torus acting on them. In this sense, they are almost toric varieties. This observation will enable us to associate data to the boundary divisors analogous to the rays in the standard fan of the previous section.

For the remainder of this section we will use I_j to denote the ideal (x, y^j) .

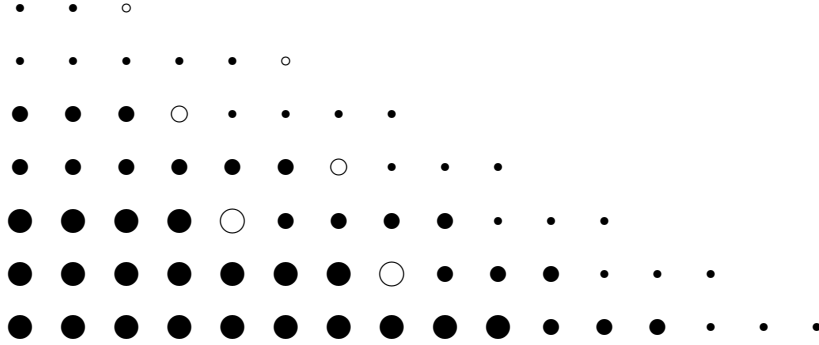
Example 4.1.

Consider the space $F(I_3, I_4, I_5)$. There is a natural projection from this space to the space $F(I_3, I_4, I_5, I_1 I_4)$ which is the projectivization of the bundle $V(I_4/I_1 I_4)$ (as defined in [11]) over $F(I_3, I_4, I_1 I_4)$. Roughly $V(I_4/I_1 I_4)$ is the bundle with fiber J_2/J_3 over a point $(J_1, J_2, J_3) \in F(I_3, I_4, I_1 I_4)$. This projection contracts the \mathbb{P}^1 of points of the form

$$(I_1^2, I_1 I_2, I_2^2 + I_1^3, (sx^2 + ty^3) + I_1^2 I_2)$$

to the point

$$c = (I_1^2, I_1 I_2, I_2^2 + I_1^3)$$

FIGURE 3. $F(3; 4; 1, 4; 5; 1, 5)$

and is a local isomorphism everywhere else. Although the \mathbb{P}^1 lying above c separated the divisors $D(0, 1)$ and $D(1, 2)$, corresponding to the rays through $(0, 1)$ and $(1, 2)$ respectively in the relevant standard fans, the point c lies on both of these divisors. From Table 2 [11], we see that the preimages of $D(0, 1)$ and $D(1, 2)$ in the fiber of J_5 over P are the loci of points of the form

$$((sx + ty^2, y^3), I_1 I_2, (uxy + vy^3, x^2, xy^2, y^4))$$

and

$$(I_1^2, (sxy + ty^2, x^2, xy^2, y^3), (u(sxy + ty^2) + vx^2) + I_1^3)$$

respectively. Since these two boundary divisors lie in planes in the Plücker embedding meeting only at c , the Zariski tangent space to c in $F(I_3, I_4, I_5)$ must have dimension at least 4. Hence the singular locus of $F(I_3, I_4, I_5)$ consists of the point c .

Example 4.2.

Consider the space

$$Y = F(I_3, I_4, I_1 I_4, I_5, I_1 I_5).$$

By Lemma 3.3 of [10], the coset representatives for $G(I_2)$ under right multiplication by $G(I_2, I_3, I_4, I_1 I_4, I_5, I_1 I_5)$ are automorphisms g with

$$g(x) = x + ay^2 + by^3 + cy^4$$

and

$$g(y) = y.$$

Thus a , b , and c form coordinates on the interior of Y . The picture P obtained by plotting the points of the exponent vectors of the non-zero terms of the coordinate functions for Y is shown in Figure 4. The x and y coordinates of the dots correspond to the exponents of a and b respectively. The sizes of the dots correspond to the exponent of c . Open circles are used for points for which the corresponding monomial is not in the span of the coordinate functions for Y . The generators for the vector space spanned by the coordinate functions modulo the space spanned by the monomials in the span are

$$a^5(ac - b^2)(ac - 2b^2)$$

and

$$a^2b(ac - b^2)(ac - 2b^2).$$

Henceforth, consider Y to be embedded in projective space with a basis of the span of the coordinate functions consisting of monomials together with these two functions as coordinate functions. There are 8 faces of the convex hull H of P , five corresponding to subvarieties of the boundary and three corresponding to the closures of the coordinate planes. Of the latter five planes, all but the one passing through the open circles correspond to boundary divisors. The one passing through the open circles corresponds to the line of points with all coordinates with monomial coordinate functions zero. There are two more boundary divisors, corresponding to the two roots of the non-monomial coordinate functions. Thus, the point whose only nonzero coordinate is the one with coordinate function $a^5(ac - b^2)(ac - 2b^2)$ is singular.

REFERENCES

- [1] Arnol'd, V.I., S. M. Gusein, A. N. Varchenko *Singularities of Differentiable Maps* Birkhauser, Boston, 1985-1988.
- [2] Briançon, J. *Description de $\text{Hilb}^n \mathbb{C}\{x, y\}$* . Inventiones math. **41**, 45–89 (1977).
- [3] Colley, S. J., G. Kennedy *The enumeration of simultaneous higher-order contacts between plane curves*. Compositio Math. **93** no. 2, 171–209 (1994).
- [4] Collino, A. *Evidence for a conjecture of Ellingsrud and Strømme on the Chow ring of $\text{Hilb}_d(\mathbb{P}^2)$* . Illinois Journal of Mathematics **32**, 171–210 (1988).
- [5] Fulton, W. *Intersection Theory*, Springer-Verlag, New York, 1984.
- [6] Fulton, W. *Introduction to Toric Varieties*, Princeton University Press, Princeton, 1993.
- [7] Iarrobino, A. *Punctual Hilbert schemes* Mem. Amer. Math. Soc. **10** no. 188 (1977).
- [8] Kleiman, S., R. Piene *Enumerating singular curves on surfaces*, Algebraic Geometry: Hirzebruch 70 (Warsaw 1998), 209–238, Contemp. Math., 241, Amer. Math. Soc, Providence, RI, 1999.

- [9] Nakajima, H. *Lectures on Hilbert schemes of points on surfaces*, University Lecture Series, **18**, Amer. Math. Soc., Providence, RI, 1999.
- [10] Russell, H. *Alignment Correspondences* (preprint)
- [11] Russell, H. *Counting Singular Plane Curves Via Hilbert Schemes* (preprint)
- [12] Russell, H. *Hilbert Schemes and Monomial Ideals*, thesis, Harvard University, June 1999.